ISSN 2614-5103 (Print)
2614-5111 (online)
IndoMath
Indonesia Mathematics Education
https://indomath.org/index.php/indomath
Vol 9, No. 2, August 2026, pp. 157-172

Junior High School Students' Conceptual Understanding of Flat-Sided Geometric Shapes through APOS Theory: A Qualitative Study

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ABSTRACT

Research on students' difficulties in geometry has mostly emphasized errors and formula mastery, while fewer studies have examined how junior high school students construct three-dimensional geometric concepts through APOS theory. This qualitative descriptive study analyzes eighth-grade students' conceptual understanding of cubes and prisms in flat-sided geometric shapes using the APOS framework. Participants were 28 eighth-grade students in Tangerang Regency who had studied cubes and prisms. Data were collected through five open-ended diagnostic questions mapped to Action, Process, Object, and Schema indicators and through semi-structured interviews. The results showed that 10 students (35.7%) were at the Action stage, 9 students (32.1%) at the Process stage, 6 students (21.4%) at the Object stage, and 3 students (10.8%) at the Schema stage. Most students could apply formulas but had difficulty explaining why formulas worked, coordinating two- and three-dimensional representations, and integrating base area, height, volume, and context. The novelty of this study lies in using APOS as a diagnostic framework for identifying were students' cognitive construction stalls in junior high school three-dimensional geometry. The findings suggest that geometry instruction should include visualization, manipulation, and reflective explanation to support transitions between APOS stages.

Keywords: APOS Theory, Conceptual Understanding, Flat-Sided Solids, Junior High School Students, Qualitative Research

ABSTRAK

Penelitian tentang kesulitan siswa dalam geometri umumnya masih menekankan kesalahan dan penguasaan rumus, sedangkan kajian yang menjelaskan bagaimana siswa SMP mengonstruksi konsep geometri tiga dimensi melalui teori APOS masih terbatas. Penelitian deskriptif kualitatif ini menganalisis pemahaman konseptual siswa kelas VIII tentang kubus dan prisma pada materi bangun ruang sisi datar dengan menggunakan kerangka APOS. Partisipan penelitian adalah 28 siswa kelas VIII di Kabupaten Tangerang yang telah mempelajari kubus dan prisma. Data dikumpulkan melalui lima soal diagnostik terbuka yang dipetakan pada indikator Action, Process, Object, dan Schema, serta melalui wawancara semi-terstruktur. Hasil penelitian menunjukkan bahwa 10 siswa (35,7%) berada pada tahap Action, 9 siswa (32,1%) pada tahap Process, 6 siswa (21,4%) pada tahap Object, dan 3 siswa (10,8%) pada tahap Schema. Sebagian besar siswa dapat menggunakan rumus, tetapi masih kesulitan menjelaskan alasan konseptual di balik rumus, mengoordinasikan representasi dua dan tiga dimensi, serta mengintegrasikan luas alas, tinggi, volume, dan konteks. Kebaruan penelitian ini terletak pada penggunaan APOS sebagai kerangka diagnostik untuk mengidentifikasi titik terhentinya konstruksi kognitif siswa dalam geometri tiga dimensi tingkat SMP. Temuan ini menyarankan pembelajaran geometri yang memuat visualisasi, manipulasi, dan penjelasan reflektif untuk mendukung transisi antar tahap APOS.

Kata Kunci: Teori APOS, Pembelajaran Geometri, Pemahaman Spasial, Kubus Dan Prisma, Penelitian Kualitatif

INTRODUCTION

Understanding geometric concepts is a crucial component of mathematics learning in schools because it plays a significant role in developing spatial thinking, logical reasoning, and problem-solving skills (Jones, 2001). Geometry requires not only calculation skills but also the ability to connect visual, symbolic, and verbal representations (Battista et al., 2018; Kinach, 2012; Weckbacher & Okamoto, 2018). One topic that forms the basis for mastering geometric concepts is plane-sided geometric shapes, particularly cubes and prisms. This topic is crucial because it lays the foundation for understanding more complex three-dimensional geometric concepts at subsequent levels of education. However, various studies have shown that students often struggle to understand the relationships between geometric elements, such as base area, height, and volume (Seah & Horne, 2020; Sudirman et al., 2023; Urbano-Urbano et al., 2021).

In classroom learning practices, the difficulty is often reinforced by formula-centered instruction. In many classroom settings, instruction tends to emphasize procedural solutions through the application of formulas without providing enough opportunities for students to visualize solids, decompose their parts, or justify why a formula represents volume or surface area (Ademmer & Prediger, 2025; Ergene & Bostan, 2025). As a result, many students understand volume and surface area solely as the result of substituting numbers in formulas, rather than as representations of three-dimensional spatial structures (Gulkilik, 2022; Safitri et al., 2023). This makes it difficult for students to explain the mathematical reasoning behind formulas or to solve new geometric contexts that require conceptual generalization (Ellis et al., 2022; Ramírez-Uclés & Ruiz-Hidalgo, 2022). This condition forms an important bridge to the research problem: students' errors are not merely computational mistakes but may reflect incomplete cognitive construction of geometric concepts (Surif et al., 2012).

This classroom phenomenon has been consistently reported in previous empirical studies investigating students' understanding of solid geometry. Previous studies have documented several related issues in solid geometry learning. Students often experience difficulties in coordinating two- and three-dimensional representations, understanding cube arrays, interpreting nets, and distinguishing volume from surface area (Battista & Clements, 1996; Chiphambo & Mtsi, 2021; Setiawati et al., 2019; Tan Sisman & Aksu, 2016; Wright & Smith, 2017). Other studies also show that students' errors in prism, pyramid, volume, and surface-area problems are frequently related to weak conceptual understanding, incorrect formula use, and limited spatial reasoning (Hasanah & Yulianti, 2020; Kurt et al., 2023; Şahin et al., 2020). Studies conducted in Indonesian junior high school contexts have further shown that spatial ability dimensions, particularly visualization, spatial relation, and spatial perception, significantly differentiate students' understanding of flat-sided solid geometry, and that students experience difficulty with higher-order geometry tasks (Lutfi & Jupri, 2020; Lutfi, Herman, & Mulyaning, 2025; Lutfi, Mulyaning, et al., 2024). These studies are valuable, but most of them emphasize errors, misconceptions, spatial ability, or representational difficulties rather than explaining how students' mental constructions develop from procedural actions into more integrated geometric schemes. This limitation is significant because teachers need to know not only what students answer incorrectly, but also at which cognitive transition their understanding breaks down, thereby enabling more targeted instructional interventions (Arnon et al., 2014c; Dubinsky et al., 2014).

Although previous studies have successfully identified students' misconceptions, spatial difficulties, and procedural errors in solid geometry, relatively little attention has been paid to the cognitive mechanisms underlying how these difficulties emerge and develop. Consequently, there remains limited understanding of the transition from procedural execution to integrated conceptual reasoning. Addressing this gap requires an analytical framework capable of explaining students' cognitive construction rather than merely describing their errors. A theoretical framework capable of explaining these cognitive transitions is therefore needed to move beyond descriptive accounts of students' errors. One relevant framework for analyzing this cognitive transition is the APOS (Action, Process, Object, Schema). This theory explains that mathematical understanding develops through four main stages: (1) Action, when students explicitly carry out procedures based on instructions; (2) Process, when students begin to mentally perform these actions and understand their interrelationships; (3) Object, when students can view concepts as manipulable entities; and (4) Schema, when various conceptual objects are integrated into more complex knowledge structures (Arnon et al., 2014a, 2014b, 2014c; Dubinsky, 2001). Because APOS focuses on the cognitive construction of mathematical concepts, it provides a suitable framework for investigating students' conceptual understanding in geometry (Firdaus et al., 2023). APOS theory has been widely used to explore students' thinking on topics such as functions, limits, and probability, but its use for diagnosing junior high school students' understanding of three-dimensional flat-sided solids remains limited. Therefore, this study differs from previous research by positioning APOS not only as a theoretical explanation, but also as a diagnostic tool for mapping students' cognitive profiles in cubes and prisms.

Based on this framework, this study aims to analyze junior high school students' conceptual understanding of flat-sided geometric shapes, specifically cubes and prisms, through the APOS theory approach (Arnon et al., 2014c; Dubinsky & McDonald, 2001). A diagnostic test with five open-ended questions was developed to represent APOS indicators, and semi-structured interviews were conducted to explore the reasoning behind students' written responses. The analysis focuses on identifying the dominant APOS stage shown by students, the difficulties they experience in moving from one stage to another, and the instructional implications of these cognitive profiles. Unlike previous studies that primarily examined geometry misconceptions, spatial ability, or representational errors, this study employs APOS theory as a diagnostic framework to identify students' cognitive transitions across the Action, Process, Object, and Schema stages (Battista, 2017; Surif et al., 2012).

This approach provides a more comprehensive explanation of how conceptual understanding develops in learning cubes and prisms (Arnon et al., 2014c; Dubinsky & McDonald, 2001).

Theoretically, this research contributes to expanding the application of APOS theory to three-dimensional geometry at the junior high school level. Practically, the results provide teachers with a more precise basis for designing learning strategies that help students move from formula use toward conceptual construction, especially through visualization, manipulation of geometric representations, and reflective explanation of the relationships among base area, height, volume, and surface area. Accordingly, this study is expected to provide both theoretical insights into students' cognitive development and practical guidance for improving geometry instruction in junior high schools.

METHOD

This study used a descriptive qualitative approach aimed at in-depth descriptions of students' cognitive development in understanding the concept of flat-sided geometric shapes, specifically cubes and prisms (Creswell & Poth, 2025; Tisdell et al., 2025). This qualitative approach was chosen because it allowed researchers to explore in detail students' thought processes in constructing geometric concepts based on the cognitive stages described in the APOS (Action, Process, Object, Schema) theory. This approach emphasizes not only learning outcomes but also how students construct mathematical knowledge through mental activity and conceptual representation. Based on this qualitative design, participants were purposively selected to provide rich information regarding students' cognitive constructions across the APOS stages (Arnon et al., 2014a, 2014b, 2014c; Dubinsky & McDonald, 2001).

The subjects in this study were 28 eighth-grade students from a junior high school in Tangerang Regency who had studied cubes and prisms in the mathematics curriculum. Subjects were selected using purposive sampling with three considerations: students had completed the relevant topic, the class represented varied mathematical achievement levels, and the participants were available to complete both the diagnostic test and follow-up interview process (Creswell & Poth, 2025). The sample size was considered adequate for qualitative inquiry because the objective was to obtain rich descriptions of students' cognitive constructions rather than statistical generalization (Creswell & Poth, 2025). Futhermore, students' ability levels were initially identified based on prior mathematics test scores and were subsequently confirmed through their diagnostic test responses. Interview participants were selected to represent the APOS categories found in the test results so that students at Action, Process, Object, and Schema stages were included in the qualitative exploration.

To explore students' cognitive constructions comprehensively, data were collected using complementary written and verbal instruments. The main instruments in this study consisted of a diagnostic test and a semi-structured interview. The use of diagnostic tasks and interviews is consistent with previous qualitative studies in geometry education that examined students' errors, reasoning, and representations in volume and surface-area problems (Chiphambo & Mtsi, 2021; Hasanah & Yulianti, 2020; Setiawati et al., 2019). The diagnostic test consisted of five open-ended essay questions designed to represent the four stages of thinking in APOS theory: Action, Process, Object, and Schema (Arnon et al., 2014a, 2014b, 2014c). Each item was connected to a specific APOS indicator so that students' written responses could be interpreted consistently. The semi-structured interview was used to clarify students' strategies, reasons for choosing procedures, and ability to connect visual, symbolic, and contextual representations (Creswell & Poth, 2025; Tisdell et al., 2025). The blueprint of the diagnostic test and coding indicators is presented in Table 1.

Table 1. APOS-Based Diagnostic Test Blueprint and Coding Indicators

APOS stage	Operational indicator	Task focus	Coding criterion
Action	Carries out an explicit procedure when dimensions or formulas are given.	Calculating the volume of a cuboid or prism and explaining the procedure.	Student applies the formula correctly or partly correctly, but the explanation remains procedural.
Process	Interiorizes the procedure and explains relationships among dimensions or shapes.	Explaining the relationship between a cuboid and a cube in deriving volume.	Student explains the transformation or relationship, but has limited generalization.
Object	Treats a solid as a whole object that can be mentally represented and manipulated.	Drawing a triangular-prism net and relating its parts to surface area.	Student coordinates two-dimensional and three-dimensional representations.
Object-Schema transition	Generalizes invariant relationships among base area, height, and volume.	Comparing two prisms with different bases but equal base area and height.	Student justifies why the volumes are equal despite different base shapes.
Schema	Integrates several geometric concepts and applies them in a new context.	Solving a contextual prism problem involving volume, surface area, and representation.	Student connects concepts flexibly and adapts the strategy independently.

Based on Table 1 summarizes the blueprint of the APOS-based diagnostic test and the coding indicators used to identify students' levels of geometric thinking in solid geometry concepts. The test is organized according to the five stages of APOS theory: Action, Process, Object, Object-Schema Transition, and Schema (Arnon et al., 2014a, 2014b, 2014c; Dubinsky, 2001; Dubinsky & McDonald, 2001). Each stage is represented through specific operational indicators, task focuses, and coding criteria that reflect the progression of students' conceptual understanding from procedural execution to integrated and flexible reasoning. At the Action stage, students carry out explicit procedures such as calculating the volume of a cuboid or prism using given formulas. At the Process stage, students begin to internalize procedures and explain relationships among geometric figures, such as relating a cube to a cuboid in deriving volume concepts.

The Object stage emphasizes students' ability to treat a geometric solid as a complete mental object that can be represented and manipulated through different forms, such as connecting prism nets with surface-area concepts (Dubinsky, 2001; Dubinsky & McDonald, 2001). The Object-Schema Transition stage assesses students' capacity to generalize invariant relationships among geometric attributes, for example by explaining why prisms with different base shapes but equal base areas and heights have the same volume. Finally, the Schema stage reflects the integration of multiple geometric concepts in solving contextual problems involving volume, surface area, and representation simultaneously.

The coding criteria were designed to classify students' responses according to the characteristics of each APOS stage. Responses classified at the Action and Process stages were characterized by procedural execution and partial conceptual understanding, whereas responses classified at the Object and Schema stages demonstrated conceptual integration, abstraction, and flexible application of geometric concepts (Arnon et al., 2014a, 2014b, 2014c). This framework enables a systematic analysis of students' cognitive development and provides evidence of their progression toward more advanced geometric understanding.

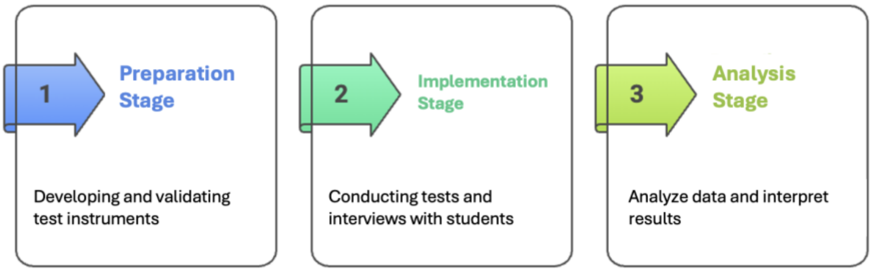

Figure 1. Research Procedure

The research procedure was carried out through three main stages as shown in Figure 1: preparation, implementation, and data analysis. The preparation stage included developing APOS-based indicators, constructing diagnostic-test items and interview guidelines, and validating the instruments with two mathematics education experts. The validation considered content relevance, item construction, language clarity, and alignment between each item and the APOS stages. Both validators judged the instrument suitable for use after minor revisions, particularly in item wording and the clarity of APOS-stage indicators. The implementation stage was carried out by administering the diagnostic test to all research subjects individually in class, followed by interviews with selected students representing the test-result categories. The analysis stage was carried out by reviewing students' written work and interview transcripts to identify thinking patterns at each APOS stage.

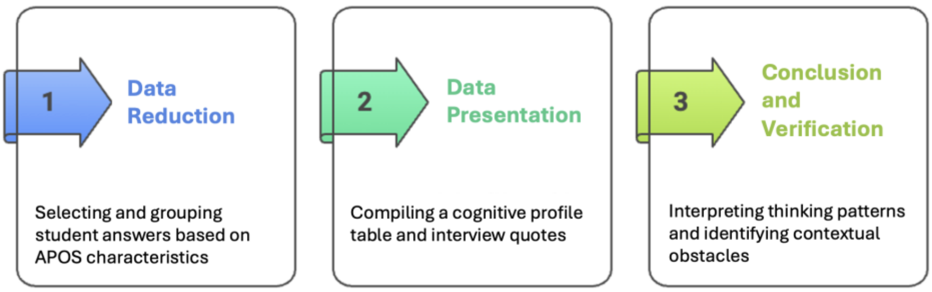

Figure 2. Miles and Huberman Model Data Analysis Technique

Data analysis was conducted using the Miles et al. (2014), as shown in Figure 2. This involved three main steps: data reduction, data presentation, and drawing conclusions and verification. In the data reduction stage, students' answers were selected, coded, and grouped using the APOS rubric in Table 1. A response was coded as action when the student only followed an explicit formula or procedure, process when the student could explain the procedure mentally and relate it to another representation, object when the student could treat the geometric concept as a complete object that could be represented or manipulated, and schema when the student could integrate several concepts in a new situation (Arnon et al., 2014a, 2014b, 2014c). The coding of written responses was discussed through peer debriefing with mathematics education researchers to maintain consistency of interpretation (Creswell & Poth, 2025; Tisdell et al., 2025). In the data presentation stage, the results were organized in tables of students' cognitive profiles and descriptive narratives supported by interview excerpts. The final stage was drawing conclusions and verification by interpreting students' thinking patterns and relating them to possible obstacles in developing conceptual schemes.

To ensure data validity, this study employed method triangulation by comparing diagnostic-test responses with semi-structured interview data (Creswell & Poth, 2025; Tisdell et al., 2025). The triangulation process was carried out in three steps. First, students' written answers were coded based on the APOS indicators. Second, interview transcripts were examined to confirm whether students could explain the reasoning behind their written procedures and representations. Third, inconsistencies between written and oral data were discussed through peer debriefing, and the final APOS classification was assigned only to the stage that was consistently supported by the available evidence. Through this procedure, the research results are expected to provide an accurate picture of students' cognitive development in understanding flat-sided geometric shapes based on APOS theory.

RESULT AND DISCUSSION

Results

Overview of Conceptual Understanding Test Results

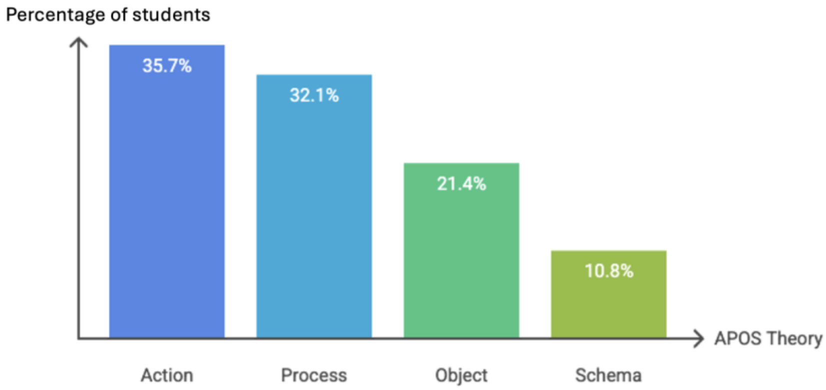

Figure 3. Percentage of student test results based on APOS Theory

Figure 3 presents the distribution of students' conceptual understanding of cubes and prisms based on the APOS cognitive stages. The results show that most students were concentrated in the lower levels of cognitive construction, namely the Action and Process stages. Specifically, 10 students (35.7%) were classified at the Action stage and 9 students (32.1%) at the Process stage, indicating that nearly two-thirds of the participants had not yet developed higher levels of conceptual understanding. In contrast, only 6 students (21.4%) reached the Object stage, while 3 students

(10.8%) attained the Schema stage. Overall, these findings suggest that although some students had progressed toward more integrated conceptual understanding, the majority remained at stages characterized by procedural execution and emerging conceptual reasoning.

Analysis of Each Question Based on Cognitive Stages

Question 1 (Action Stage)

Students were asked to calculate the volume of a cuboid with certain dimensions. Most students were able to use the formula $V = p \times l \times t$, but they were unable to explain why multiplying length, width, and height represents volume. Interview excerpt (S1):

"I multiply the length, width, and height because that's the formula, ma'am. I don't know why it's like that."

This response is consistent with the Action indicator because the student depended on an external formula and did not yet show mental reconstruction of the spatial meaning of volume.

Question 2 (Process Stage)

Students were asked to explain how the relationship between a cube and a cuboid can be used to derive the formula for the volume of a cube. Some students began to transform the representation from a cuboid to a cube by recognizing that the three dimensions become equal. Interview excerpt (S7):

"If all sides of a cuboid are equal, then the length, width, and height are equal, so the formula is $V = s \times s \times s$ or $V = s^3$."

This response demonstrates the Process stage because the student could explain the relationship among dimensions and mentally transform a cuboid into a cube, although the explanation had not yet developed into broader conceptual generalization.

Question 3 (Object Stage)

Students were asked to draw the net of a triangular prism and explain its relationship to the surface area of the prism. Students who reached this stage were able to manipulate objects mentally. Interview excerpt (S13):

"If you unfold it, it has three rectangles and two triangles. So, you just add them all up."

This response is consistent with the Object stage because the student treated the prism as a complete geometric object that could be unfolded, represented as a net, and reconnected to the surface-area concept.

Question 4 (Object–Schema Transition)

Students were asked to compare the volumes of two prisms with different bases and heights but the same base area. Students who have reached this stage were able to generalize the concept. Interview excerpt (S22):

"As long as the base area and height are the same, the volume must be the same, even if the base shapes are different."

This response indicates an Object-Schema transition because the student generalized that volume depends on the relationship between base area and height, not merely on the visual form of the base.

Question 5 (Schema Stage)

Students were asked to solve contextual problems involving the application of the volume and surface area of prisms in real-life situations such as calculating the volume of water in a prism-shaped tank. Three students (10.8%) were able to integrate various concepts and adapt their strategy to the context. Interview excerpt (S27):

"I calculated the area of the base first using a triangle, then multiplied it by the height. Because the shape was unusual, I drew it first to understand its shape."

This response demonstrates the Schema stage because the student coordinated visual representation, base-area calculation, prism height, and contextual interpretation in one coherent solution strategy.

Student Cognitive Profiles Based on APOS Stages

Table 2. Student Cognitive Profiles Based on APOS Stages

APOS stage	Main indicator in student responses	Dominant test evidence	Students n (%)
Action	Performs procedures without explaining conceptual meaning.	Substitutes dimensions into $V = l \times w \times h$.	10 (35.7%)
Process	Explains steps and simple relations, but has limited generalization.	Relates cube and cuboid volume formulas.	9 (32.1%)
Object	Represents and manipulates solids mentally.	Draws prism nets and relates faces to surface area.	6 (21.4%)
Schema	Integrates several geometric concepts in a new context.	Connects base area, height, volume, and contextual representation.	3 (10.8%)

Based on Table 2, the distribution of students' cognitive profiles based on the APOS framework. The results indicate that most students were classified at the lower levels of APOS development. The largest proportion of students (35.7%) was categorized at the Action stage, characterized by the ability to perform procedural calculations using given formulas without demonstrating an understanding of the underlying concepts. Evidence of this stage was observed when students directly substituted dimensions into the volume formula ($V = l \times w \times h$) without providing conceptual explanations.

A further 32.1% of students reached the Process stage, where they were able to explain calculation steps and identify simple relationships among geometric concepts, such as the connection between cube and cuboid volume formulas. However, their reasoning remained limited in terms of generalization and abstraction. Meanwhile, 21.4% of students were classified at the Object stage, demonstrating the ability to mentally represent and manipulate three-dimensional figures. These students successfully drew prism nets and connected the components of the nets to surface-area concepts.

Only 10.8% of students achieved the Schema stage, the highest level identified in this study. Students at this stage integrated multiple geometric concepts simultaneously, connecting base area, height, volume, and contextual representations to solve unfamiliar problems. Overall, the findings suggest that while some students demonstrated advanced conceptual understanding, the majority remained at the Action and Process stages, indicating that procedural knowledge was more dominant than integrated conceptual reasoning in learning solid geometry.

Discussion

The results of this study indicate that most students remained at the Action and Process stages, accounting for 19 out of 28 students (67.8%). This distribution suggests that students' understanding of flat-sided geometric shapes was still largely characterized by procedural reasoning rather than conceptual understanding. Although a smaller proportion of students reached the object stage (21.4%) and the schema stage (10.8%), these findings indicate that only a limited number of students were able to construct an integrated and coherent understanding of geometric concepts. According to APOS theory, this pattern reflects that many students had not yet progressed from performing mathematical actions to developing internalized processes and subsequently encapsulating those processes into stable mathematical objects (Arnon et al., 2014b; Dubinsky, 2001). Recent studies based on APOS theory suggest that conceptual understanding develops progressively through the transformation of Action into Process, followed by the encapsulation of Process into Object, and finally the organization of these objects into a coherent Schema. Students who remain at the Action and Process stages generally have not yet established sufficiently integrated cognitive structures to support flexible mathematical reasoning and conceptual problem solving (Bansilal et al., 2017; Borji et al., 2024).

From the perspective of APOS theory, the predominance of students at the Action and Process stages indicates that learning was still focused on executing procedures with limited opportunities for reflective abstraction. Consequently, many students repeatedly applied formulas or algorithms without fully internalizing the underlying mathematical relationships. As a result, their mathematical thinking remained dependent on external procedures instead of developing into flexible cognitive structures that support conceptual reasoning, abstraction, and the coordination of interconnected mathematical ideas. This interpretation is consistent with recent APOS-based studies showing that interiorization, encapsulation, and schema construction are essential mechanisms for developing meaningful mathematical understanding rather than procedural competence alone (Cetin & Dubinsky, 2017; Tsafe, 2024; Akarsu, 2022).

These findings also reveal qualitative differences in students' conceptual understanding across the APOS stages. Students at the Action stage primarily exhibited instrumental understanding, relying on memorized procedures and formulas without adequately explaining the underlying geometric relationships. As they progressed to the Process and Object stages, students began to coordinate multiple representations, recognize relationships among geometric properties, and mentally manipulate geometric ideas, indicating a transition from procedural execution toward conceptual reasoning. Students who reached the Schema stage demonstrated a more integrated understanding by flexibly connecting concepts of volume, surface area, geometric representations, and contextual information to solve unfamiliar problems. This progression suggests that understanding cubes and prisms develops through the gradual construction of interconnected cognitive structures rather than through procedural competence alone. As these cognitive structures become more integrated, students are better able to explain mathematical relationships, generalize concepts to new situations, and apply their knowledge flexibly in solving geometric problems. Similar findings have been reported in recent studies, which emphasize that meaningful mathematical learning is characterized by the integration of conceptual knowledge, multiple representations, and flexible reasoning rather than the mere execution of algorithms (Torres-Peña et al., 2025; Widdiharto et al., 2025). This progression is consistent with APOS theory, which posits that conceptual understanding emerges through the successive processes of interiorization, coordination, encapsulation, and schema construction. As students move across these stages, their mathematical thinking becomes increasingly organized, enabling them to reason flexibly, establish connections among concepts, and solve unfamiliar problems more effectively (Bansilal et al., 2017; Borji et al., 2024).

This finding is also consistent with previous studies grounded in APOS theory, which indicate that learners commonly rely on procedural knowledge before progressing toward object-level understanding and integrated schema construction (Arnon et al., 2014b). Within the APOS framework, conceptual understanding develops through successive cognitive transitions from Action to Process, Object, and finally Schema. The predominance of students at the Action and Process stages therefore suggests that many participants had not yet fully interiorized mathematical actions into processes or encapsulated those processes into coherent mathematical objects. Similar patterns have been reported in APOS-based mathematics education research, where students demonstrate procedural competence but continue to experience difficulties in conceptual abstraction and structural reasoning. These findings are further supported by geometry education studies showing that students often struggle to connect formulas with spatial structures, nets, and three-dimensional representations (Battista & Clements, 1996; Chiphambo & Mtsi, 2021; Setiawati et al., 2019; Tan Sisman & Aksu, 2016; Wright & Smith, 2017).

One reason many students remained at the Action stage is the strong influence of formula-oriented learning. Students tended to remember that volume is obtained by multiplying length, width, and height, but they could not explain the spatial meaning of this multiplication. This suggests that their learning experience may have emphasized rule execution more than conceptual reconstruction. Such a pattern is consistent with Alghadari et al. (2024), Matheou Argyrou & Panaoura (2021), and Noor & Alghadari (2021), who showed that geometry difficulties often emerge when students cannot connect procedural steps with the concepts represented by those steps.

Students at the Process stage showed better reasoning because they could describe relationships among dimensions, such as the relationship between cuboids and cubes. However, many of them still struggled to generalize these relationships to unfamiliar objects or contextual problems. This indicates that the procedure had begun to be interiorized, but it had not yet become a flexible object of thought. The obstacle may be related to limited opportunities to explain, compare, and justify geometric relationships during instruction, as well as limited exposure to tasks that require students to move between visual, symbolic, and verbal representations.

Students who reach the Object stage are able to view geometric shapes as complete entities that could be mentally manipulated. Their ability to draw prism nets and connect the parts of the net to surface area indicates objectification of the geometric concept. From the perspective of APOS theory, this transition reflects the process of encapsulation, in which previously internalized procedures are transformed into complete mathematical objects that can be manipulated, compared, generalized, and connected with other geometric concepts (Arnon et al., 2014a; Dubinsky & McDonald, 2001). For example, students no longer viewed volume merely as the result of multiplying length, width, and height, but understood it as a geometric property that could be interpreted across different solids and representations. This finding aligns with Lavicza et al. (2023) and Yao (2022), who found that visual representations and exploratory activities can facilitate objectification in geometry learning. The result also suggests that students need repeated experiences in unfolding, rotating, comparing, and reconstructing solids so that a shape is not perceived merely as a static drawing.

The three students who reached the Schema stage demonstrated the strongest conceptual integration in this study. They could coordinate volume, base area, height, surface representation, and contextual information to solve non-routine tasks. This ability reflects the formation of a flexible cognitive structure in which several objects and processes are connected into one scheme. Nevertheless, the small percentage of students at this stage (10.8%) shows that schema construction in three-dimensional geometry is demanding. It requires not only knowledge of formulas, but also spatial visualization, representational fluency, and reflective reasoning (Chan & Kwan, 2021; Cui et al., 2024).

The progression observed across the four APOS stages illustrates that conceptual understanding develops as a continuous cognitive transformation rather than as a sequence of isolated competencies. Students at the Action stage depended on externally provided procedures, whereas those at the Process stage began to internalize these procedures into coordinated mental operations. Only after successful encapsulation were students able to manipulate geometric concepts as complete objects, eventually integrating multiple objects and processes into coherent schemas for solving unfamiliar problems. The decreasing proportion of students across successive stages therefore reflects increasing cognitive complexity rather than merely differences in mathematical achievement (Arnon et al., 2014c; Dubinsky & McDonald, 2001).

From the perspective of conceptual understanding, these findings indicate clear differences in the quality of students' mathematical knowledge across the APOS stages. Students at the Action stage primarily demonstrated instrumental understanding, relying on memorized procedures and formulas without fully explaining the underlying geometric relationships. As students progressed to the Process and Object stages, they began to coordinate multiple representations and recognize structural relationships among geometric concepts. At the Schema stage, students demonstrated relational understanding by flexibly integrating volume, surface area, geometric representations, and contextual information to solve unfamiliar problems. This progression suggests that conceptual understanding develops not only through acquiring procedural competence but also through the construction of increasingly interconnected mathematical structures, consistent with theories of relational understanding and mathematical proficiency (Kania et al., 2025).

Overall, these results emphasize the need for geometry learning that supports mental reconstruction and reflective activity. Students should be guided to explain why formulas work, compare different solids with equivalent measures, construct and deconstruct nets, and interpret geometric situations in real contexts. Dynamic visual media, manipulatives, GeoGebra 3D, and Augmented Reality (AR) can support these processes because students can manipulate objects directly and observe relationships among dimensions (Cetintav & Yilmaz, 2023; Lutfi et al., 2022; Morales Méndez & Lozano Avilés, 2025; Muhtadi & Saleh, 2020). This aligns with Battista et al. (2018), Gilligan-Lee et al. (2022), and Lakin et al. (2024) who emphasized that spatial reasoning is a key component in the development of geometric understanding.

Furthermore, the APOS theory-based approach is useful for diagnosing students' cognitive profiles and designing adaptive instruction. By knowing whether a student is at Action, Process, Object, or Schema stage, teachers can provide more appropriate interventions. Students at Action stage need concrete manipulation and explanation of formula meaning; students at Process stage need tasks that promote comparison and generalization; students at Object stage need opportunities to coordinate multiple representations; and students approaching Schema stage need complex contextual problems that require integration of several concepts. Thus, this study extends previous APOS-based research by demonstrating how students' conceptual understanding of flat-sided geometric shapes develops across the Action, Process, Object, and Schema stages within the context of junior secondary geometry. Rather than merely identifying students' misconceptions or procedural errors, the findings explain these learning difficulties as cognitive transitions that can inform more targeted instructional interventions. Consequently, APOS theory serves not only as an analytical framework for diagnosing students' conceptual understanding but also as a practical guide for designing learning experiences that promote progression toward higher levels of mathematical thinking.

CONCLUSION

This study analyzed junior high school students' conceptual understanding of cubes and prisms using the APOS theoretical framework. The results showed that students' cognitive profiles were distributed across four APOS stages: 10 students (35.7%) at Action, 9 students (32.1%) at Process, 6 students (21.4%) at Object, and 3 students (10.8%) at Schema. These findings indicate that most students were still able to use formulas procedurally but had difficulty explaining the conceptual meaning behind them, while only a small group could integrate several geometric concepts in contextual problem solving. The main contribution of this study is the use of APOS theory as a diagnostic framework for identifying where students' conceptual construction of flat-sided solids develops or stalls.

The implications of these findings suggest that geometry instruction needs to emphasize exploratory, manipulative, and reflective activities. Mathematics teachers need to facilitate students' transitions between APOS stages by asking students to justify formulas, connect nets with three-dimensional solids, compare equivalent volume situations, and use interactive visual media such as

GeoGebra 3D or Augmented Reality (AR). Such learning experiences can help students build stronger spatial representations and more coherent conceptual understanding.

Because this study used a qualitative design with 28 students from one school context, the findings should not be overgeneralized to all junior high school students. Future research is recommended to involve more participants from diverse school contexts and expand the scope of material to other geometric shapes, such as pyramids and cylinders. Further studies could also develop and test APOS-based learning models supported by digital technology to examine their effectiveness in improving students' spatial ability and conceptual understanding.

DECLARATION OF THE USE OF AI

During the preparation of this work, the authors used Grammarly to assist with language editing, grammar correction, and the enhancement of the manuscript's clarity and coherence to ensure it met academic writing standards. After using this tool, the authors reviewed and edited the content as needed. The authors take full responsibility for the content of the published article.

ACKNOWLEDGEMENT

We would like to thank the teachers, students, and all participants who were willing to take part in this study.

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